

On Smith-Volterra-Cantor sets and their measure

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Abstract

This paper analyzes common definitions of the Cantor Middle Thirds set and Smith-Volterra-Cantor sets in general (of which the Cantor Middle Thirds set is a special case) and demonstrates that there are implicit assumptions associated with such definitions which are logically unsustainable.

Version History:

Version 2: This version clarifies that if set membership requires the application of a limiting condition, then that limiting condition should be included as part of the set definition. References are now included to alternating series explaining how discrepancies of measures of Smith-Volterra-Cantor sets can arise (Section 4.4).

Version 3: Corrections to Section 1.4, and definitions 2.1, 2.2, A.8, A.9.

Version 4: Rewritten to give better exposition of limit states in general for sets defined in terms of infinitely many intervals. Added preliminary sections on sets defined in terms of infinitely many intervals.

Version 5: Sections 4.2 and 4.3 minor changes to limit expressions. Section 4.4 revised regarding the application of methods for denumerable sets to non-denumerable sets. Correction to end of Section 4.1.

Version 6: Fixed missing exponents in equations in Sections 1 and 2.

1 Definitions of sets in terms of infinitely many intervals

First we consider an informal definition of a set in terms of infinitely many open intervals. Given the interval $(0, 1)$, and an enumeration function of the rationals over this interval and which assigns a unique rational to each natural number, we define a set A by the addition of intervals to the set as follows:

Let the first rational in the enumeration be the midpoint of a open interval of measure $2/10$. Let the second rational in the enumeration be the midpoint of a open interval of measure $2/100$. And so on, so that the n^{th} rational in the enumeration is the midpoint of a open interval of measure $2/10^n$. We might try to give a formal definition of this as:

Definition 1.1. For $r \in \mathbb{R}, 0 \leq r \leq 1$

$$r \in A \iff \left\{ \exists n \in \mathbb{N} \wedge \left[q(n) - \frac{1}{10^n} < r < q(n) + \frac{1}{10^n} \right] \right\}$$

where $q(n)$ is a function that enumerates all rationals between 0 and 1 (such as in [Appendix D](#)), and its complement B as:

Definition 1.2. For $r \in \mathbb{R}, 0 \leq r \leq 1$

$$r \in B \iff \neg \left\{ \exists n \in \mathbb{N} \wedge \left[q(n) - \frac{1}{10^n} < r < q(n) + \frac{1}{10^n} \right] \right\}$$

Given this definition, there are only two possibilities. Either:

1. The entire closed interval between 0 and 1 is covered by the intervals, or
2. There are some irrational points r between 0 to 1 that are not covered by any interval (by definition there cannot be any rational points that are not covered by some interval).

If the second option is the case, any points that are not in the set A must be irrational points that are degenerate single point intervals, since any non-degenerate interval must contain rationals. Let us suppose that there are points not in the set A , any such points constitute the set B .

From the given definition, there cannot be any rational that is a degenerate single point interval, since every rational has an associated enumeration number n and hence is the midpoint of an interval of non-zero measure $2/10^n$.

Since some such intervals overlap, we will use the term “complete interval” to refer to an interval of A if it is not a sub-interval of any interval of A , apart from itself. It follows that the set A is either the complete interval $[0, 1]$ or is comprised of more than one such complete interval. For any point in the set B , there must be a complete interval of A that has this point as a left endpoint, and another complete interval of A that has this point as a right endpoint (the points that are the lower and upper bounds of the complete intervals).

We now have a contradiction, since any such point of B must be irrational, and yet it is also an endpoint of an interval that is defined to be rational.

The conventional position is that this contradiction is to be ignored, or else vaguely attributed to some aspect of the infinitely decreasing nature of the defined intervals. But without a comprehensively developed logical explanation, such an approach fails to give a satisfactory resolution of the contradiction, and fails to convince that further critical analysis is not required.

It is also conventionally asserted that the total measure of the set A cannot be more than $2/9$. This number is calculated according as the limiting value of the sum of $2/10 + 2/100 + 2/1000 + \dots$. This is a maximum value; since some intervals overlap the limiting sum will be less than $2/9$. This means that the measure of the set B must be more than $7/9$.

But it is easy to show that the points of the set B cannot possibly give rise to a total measure greater than the total measure of the set A , as follows:

Every rational has an associated n by the enumeration of the rationals. And every rational is in some complete interval of A . For any given rational, either it is the rational with the smallest associated n of all the rationals in that complete interval or it is not. If it is, then there is a unique association of that natural number n with the left irrational endpoint of that complete interval of A . Since every rational is enumerated, every complete interval of A is included by such an association, and hence every left endpoint of a complete interval of A is accounted for by such an association. Similarly, there is also a unique correspondence of the lowest associated n of all the rationals in a complete interval with the right endpoint of that interval.

Hence each element of the set of the left irrational endpoints has a unique association with an element of a subset of the natural numbers (the smallest n 's of each complete interval

of A), and hence the set of such points of B cannot constitute a set that has “more” elements than the set of natural numbers. Clearly these endpoints of the complete intervals of A , all of which are of zero measure, cannot constitute a greater measure than the intervals for which they are endpoints, since all those intervals are of non-zero measure.

Hence it is seen that the conventional position regarding the definition of the set A results in not one, but two clear contradictions that are afforded no resolution by conventional mathematics.

2 Resolution of the contradictions

The remainder of this paper will give by logical analysis a complete explanation of the causes of these contradictions and how a complete resolution of the difficulties can be resolved.

The above definition of the set A describes a summation over infinitely many intervals that has no termination, since the intervals continually decrease, but the definition does not define a state where the intervals become zero measure degenerate single points. We can also observe that the definition does not provide any definition of a state where the endpoints of the interval become irrational rather than rational.

But it is a common matter in mathematics to consider the limit state of a non-terminating series or summation. It is trivial that irrationals are definable in terms of a limit of a summation over a series of infinitely many rationals. And by an argument similar to that in Cantor’s 1874 uncountability proof,^[3] by an enumeration of rationals, a limiting value can be generated from a defined sequence that is a single value that is not a rational number.^A

2.1 Set definition in terms of infinitely many closed intervals

To analyze the situation, it is informative to first consider the case where the intervals are defined as above, except that for this case they are *closed* intervals, where the endpoints are included within the interval. We will call the set described by this case as the set A^* , and we define it in terms of limiting conditions:

Definition 2.1. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}, n > 0$,

$$r \in A^* \iff \lim_{n \rightarrow \infty} \bigcup \left\{ r : q(n) - \frac{1}{10^n} \leq r \leq q(n) + \frac{1}{10^n} \right\}$$

and its complement B^* as:

Definition 2.2. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}, n > 0$,

$$r \in B^* \iff [0, 1] - \lim_{n \rightarrow \infty} \bigcup \left\{ r : q(n) - \frac{1}{10^n} \leq r \leq q(n) + \frac{1}{10^n} \right\}$$

Consider the limit state of the closed intervals. The limit state of closed intervals of decreasing size is as a degenerate single point interval where the endpoints are coincident. For the case considered here, there are infinitely many such limit states. We can observe that no such degenerate single points can be rational numbers, otherwise for any such point, there would already be some n such that the n^{th} number in the enumeration would be that number, and hence it would be the midpoint of an interval that has, by definition, non-zero measure, namely

^A Cantor’s proof uses an assumption of the existence of an enumeration of real numbers in order to prove the converse.

$2/10^n$, rather than the midpoint of a degenerate interval of zero measure. Hence each such limit point is a Dedekind cut where all the rationals are either smaller or larger than it.

It also follows that the limit state of the summation of the intervals includes all the defined non-zero measure intervals for each rational, and all these limit state single point degenerate intervals, and hence for a definition of the set A^* that is correctly formulated with explicit inclusion of the limit states, the set A^* is the complete interval $[0, 1]$.

By a similar consideration, the remaining intervals are open intervals. The limit state of open intervals is also when the endpoints coincide, but for open intervals of decreasing measure, the intervals disappear and the limit state is as open intervals of zero measure. While the term “degenerate” is commonly used to denote a single point closed interval, it might be better to use the term “degenerate open interval” to denote the *position* of such an empty open interval whose left-side endpoint and right-side endpoint are coincident, and the term “degenerate closed interval” to denote a single point closed interval.

We can observe that the limit states of the removed intervals and the limit states of the remaining intervals are both in accordance with the above conclusion that there are no intervals remaining in the set B^* .

2.2 Set definition in terms of infinitely many open intervals

We now consider the set A as previously described in Section 1, but this time we define it with an explicit inclusion of the limit state as:

Definition 2.3. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}, n > 0$,

$$r \in A \iff \lim_{n \rightarrow \infty} \bigcup \left\{ r : q(n) - \frac{1}{10^n} < r < q(n) + \frac{1}{10^n} \right\}$$

and its complement B as:

Definition 2.4. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}, n > 0$,

$$r \in B \iff [0, 1] - \lim_{n \rightarrow \infty} \bigcup \left\{ r : q(n) - \frac{1}{10^n} < r < q(n) + \frac{1}{10^n} \right\}$$

In this case, the remaining intervals are closed intervals, and the limit state of these intervals are, as in Section 1, single degenerate intervals where the endpoints are coincident. And as in Section 1, these points cannot be rational, and must be irrational. So in this case, the set B consists of single irrationals, and these points are not denumerable.

This does not mean that there are in some sense “more” points in the set A than the set B , since, as pointed out in Section 1, every such point r is uniquely associated with the natural number that enumerates the midpoint of the largest interval within the complete interval of A of which r is the endpoint. The non-denumerability is because the limit state is never actually reached at any point in the continuing decrease of the width of the intervals, and there is no definable transition from the infinitely many decreasing intervals to the infinitely many limit state points.^B

In summary, by explicitly including the limit condition in the definition, the contradictions generated by naive definitions such as 1.1 and 1.2 disappear.

^B Note the difference between these cases which involve infinitely many limit states, and single limit state cases such as an irrational number that is given by the limit of a summation of a single series.

3 The Cantor Middle Thirds Set

We now analyze the Cantor Middle Thirds Set, hereinafter referred to simply as the Thirds set. The Thirds set is a special case of Smith-Volterra-Cantor sets in general, which are addressed in Section 4. A typical treatment of the subject of Thirds sets and Smith-Volterra-Cantor sets is given by Vallin.^[14]

3.1 Definitions of a Cantor middle thirds set

A typical informal description of a Cantor Middle Thirds set is as follows:

Definition 3.1. Given the closed real interval $[0, 1]$, perform the following recursive process:

The first iteration is the removal of the central open middle third of the interval $[0, 1]$. For all subsequent iterations, remove the central open middle third of each interval in the set of all numbers between 0 and 1 (including 0 and 1) that is the result of the previous iteration. The Cantor Middle Thirds set is the set that remains from infinitely many repetitions of this recursion.

Another definition which is commonly used is given by:^{[5],[Ch.5]}

Definition 3.2.

T_0 is the interval $[0, 1]$,
 T_1 is the interval $[0, 1]$ less the interval $(1/3, 2/3)$,
 T_2 is the interval $[0, 1]$ less the intervals $(1/9, 2/9)$, $(1/3, 2/3)$ and $(7/9, 8/9)$,
and so on. The Thirds set is then:

$$\bigcap_{n=1}^{\infty} T_n \quad \text{where } T_n \text{ is the set of all intervals remaining after the } n^{\text{th}} \text{ iteration.}$$

A more detailed formal definition, but which is based on the union of removed open intervals, is given by:^[11]

Definition 3.3.

$$[0, 1] - \bigcup_{n=1}^{\infty} \bigcup_{k=0}^{3^{n-1}-1} \left(\frac{3k+1}{3^n}, \frac{3k+2}{3^n} \right)$$

In terms of the iterative removal of the middle third of each interval, the first two iterations are illustrated diagrammatically below:



It will be noted that none of the above definitions make any reference to a limit state, even though they all refer to the set as a summation over infinitely many intervals.

Conventionally, the Thirds set is considered to consist of elements whose ternary expansions are in accordance with the following rules:

Thirds Conjecture: Elements of the Thirds set

In terms of ternary expansions, the elements of the Thirds set can be defined in terms of finite or infinite expansions using only the digits 0, 1 and 2. As with an expansion in any base, some numbers will have two different ternary expansions, for example, the ternary expansion $0.02222\dots$ has the same numerical value as the ternary expansion 0.1 and the ternary expansion $0.12222\dots$ has the same numerical value as the ternary expansion 0.2 .

A number is considered to be an element of the Thirds set provided there exists a ternary expansion of that number that does not include the digit 1 anywhere in the expansion (i.e., if there are two ternary expansions and provided that only one expansion includes the digit 1, then the number is in the Thirds set). A number is considered not to be a member of the Thirds set if it has a singular ternary expansion that has a 1 somewhere in the expansion, or if it has dual ternary expansions where both have a 1 somewhere in the expansion. For example: $1/3$ has two ternary expansions, 0.1 and $0.0222\dots$ and is considered to be in the Thirds set; $2/3$ has two ternary expansions, 0.2 and $0.1222\dots$, and is considered to be in the Thirds set; $4/9$ has two ternary expansions, 0.11 and $0.10222\dots$, and is considered to be not in the Thirds set.

The standard justification for the above is by reference to first-order definitions,^C where the Thirds set is considered to be defined in terms of the subtraction of the summation of all the removed intervals after the n^{th} iteration:^[11]

^C Following the principle of the definition 3.3.

Definition 3.4. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}$

$$r \in T_n \iff \neg \left\{ \exists m \in \mathbb{N}, 0 < m \leq n, \exists k \in \mathbb{N}, 0 \leq k \leq (3^{m-1} - 1) \wedge \frac{3k+1}{3^m} < r < \frac{3k+2}{3^m} \right\}$$

and the complementary set by:

Definition 3.5. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}$

$$r \in U_n \iff \left\{ \exists m \in \mathbb{N}, 0 < m \leq n, \exists k \in \mathbb{N}, 0 \leq k \leq (3^{m-1} - 1) \wedge \frac{3k+1}{3^m} < r < \frac{3k+2}{3^m} \right\}$$

Each set U_n corresponds to the set of points removed after the n^{th} iteration, while each set T_n corresponds to the set of points remaining after the n^{th} iteration. Given the above definitions, the justification for the [Thirds Conjecture](#) regarding the elements of the Thirds set is as follows, where each x can be any of the digits 0, 1, or 2, except where specified:

- 3.1.a) For $n = 1$, any number with an expansion of the form: $0.1xxxx\dots$, where not all $x = 0$ and not all $x = 2$, is defined by the definitions [3.5](#) and [3.4](#) to be in the set U_1 , and not in the set T_1 ; hence such numbers are not in the Thirds set.
- 3.1.b) For $n = 2$, any number with an expansion of the form: $0.01xxxx\dots$ or $0.21xxxx\dots$, where not all $x = 0$ and not all $x = 2$, is defined by the definitions [3.5](#) and [3.4](#) to be in the set U_2 , and not in the set T_2 ; hence such numbers are not in the Thirds set.
- 3.1.c) For $n = 3$, any number with an expansion of the form: $0.001xxxx\dots$, $0.021xxxx\dots$, $0.201xxxx\dots$ or $0.221xxxx\dots$, where not all $x = 0$ and not all $x = 2$, is defined by the definitions [3.5](#) and [3.4](#) to be in the set U_3 , and not in the set T_3 ; hence such numbers are not in the Thirds set.
- 3.1.d) In general, any number with an expansion that has a 1 at the n^{th} digit of the ternary expansion is defined by the definitions [3.5](#) and [3.4](#) to be in the set U_n , and not in the set T_n , provided that all subsequent digits of the expansion are neither all 0 nor all 2. Alternatively, this can be stated as: Taking the ternary expansion that ends in an infinitely repeating sequence of 2s rather than the expansion that is finite and ends with the digit 1 or 2, then any number with an expansion that has a 1 at the n^{th} digit of the ternary expansion is defined by the definitions [3.4](#) and [3.5](#) to be in the set U_n , and not in the set T_n . Hence, if there are two ternary expansions and provided that only one expansion includes the digit 1, then the number is in the Thirds set.

Conventional descriptions^D of the Thirds set assume that a definition such as [3.4](#) above gives a definitive description of the elements of the Thirds set, since it is defined in terms of infinitely many iterative removals of middle third closed intervals.

Assuming the excluded middle, every real number between 0 and 1 must belong to either the Thirds set or its complement. In the set that satisfies the [Thirds Conjecture](#), there cannot be any non-degenerate interval that consists entirely of numbers for which there is a ternary expansion that does not include any digit as 1, since given any two such numbers, there are infinitely many numbers between two such numbers for which there is a ternary expansion that includes the digit 1. Hence the set satisfying the [Thirds Conjecture](#) consists of isolated points.

^D See references [\[1\],\[2\],\[4\],\[6\],\[7\],\[9\],\[12\],\[13\]](#)

It can be readily demonstrated^D that the set described by the [Thirds Conjecture](#) is not denumerable, see [Appendix C](#). It is commonly observed that this is a counter-intuitive result, since on the one hand, the iterations are defined to be denumerable, and the intervals that result from the iterations are also defined to be denumerable, yet it is asserted that such iterations define a set that consists of isolated points that are not denumerable. This is a paradoxical anomaly for which no satisfactory explanation has been presented. But it should matter that expositions that yield paradoxical results are commonly accepted as correct, whereas any such anomalies should be subjected to rigorously logical analysis.

3.2 Fixed ratio dividing points

Given the notion of iterative removal of intervals that can be associated with the above definitions, we can determine, in general, the numerical values of points that avoid removal by any n^{th} iteration of the process. We observe that the numerical value of such points is such that they must divide every interval that they occur within, in precisely the same ratio. This means that if such a point remains after one iteration, it must remain after any subsequent iterations, since in terms of ratios, each iteration performs precisely the same process on any remaining interval. The existence of such points are determined by considering if there are any points which divide a given interval in a certain ratio, and where a remaining interval given by the subsequent iteration is such that:

- (i) the point is within that interval and
- (ii) the point divides that interval by precisely the same ratio.

There are two ways in which this can occur; given such a ratio, either the division is such that:

$$\frac{\text{left-side division of the interval}}{\text{right-side division of the interval}} = \frac{a}{b} \text{ for all iterations, or}$$

for every alternate iteration, the ratio is reversed, i.e.,

$$\text{for } n = 1, 3, 5, \dots, \frac{\text{left-side division of the interval}}{\text{right-side division of the interval}} = \frac{a}{b}$$

$$\text{for } n = 2, 4, 6, \dots, \frac{\text{right-side division of the interval}}{\text{left-side division of the interval}} = \frac{a}{b}$$

Since all subsequent iterations are self-similar, such a point will also divide all subsequent intervals which it is within by precisely the same ratio. To find such points, we suppose that there is such a point x in an open interval $(0, q)$, and which is such a point in the subsequent left-side open interval $(0, \frac{q}{3})$. Then we could have either:

- (i) if the ratio does not alternate at each iteration:

$$\frac{q-x}{q} = \frac{\frac{q}{3} - x}{\frac{q}{3}}$$

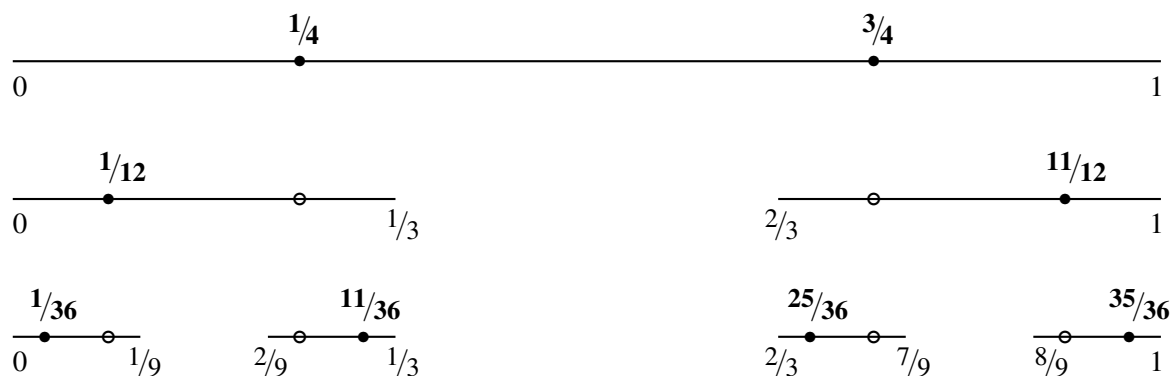
- (ii) or if the ratio alternates direction at each iteration:

$$\frac{q-x}{q} = \frac{x-0}{\frac{q}{3}}$$

Simplifying, the first gives the value $x = 0$, which is trivially an endpoint, while the second gives $x = q/4$ i.e., for $q = 1$ this gives the point $x = 1/4$. By symmetry, or by a similar analysis,

for the subsequent right-side interval $(\frac{2}{3}, 1)$, the relevant points are $x = 1$, again trivially an endpoint, and $x = \frac{3}{4}$.

As noted above, all subsequent iterations are self-similar, so that each iteration will also produce further such points that are such so as to always divide every subsequent interval in precisely the same ratio. Since that is the case, there cannot be any n^{th} iteration that removes such points. It will also be noted that there is a new fixed ratio dividing point for each new remaining interval produced at each iteration, see the diagram below.



It will be noted that the set of fixed ratio dividing points is a set of points that is defined in terms of a reference to the endpoints $\left[\frac{3k+1}{3^m}, \frac{3k+2}{3^m} \right]$. For convenience, these fixed ratio dividing points are henceforth termed FRD points. In general such points are *included* among the values:

$$\frac{3k+1 - \frac{1}{4}}{3^m} \quad \text{and} \quad \frac{3k+2 + \frac{1}{4}}{3^m}, \quad \text{where } 0 \leq k \leq (3^{m-1} - 1).$$

but note that these formulas also include points within intervals removed by previous iterations.

These FRD points are denumerable. They are all rational numbers since they are generated by defining a ratio of the difference of two rational numbers; they all have a non-terminating ternary expansion that consists only of the digits 0 and 2, and for any n^{th} iteration, from some point in the sequence of digits the number consists of an infinitely repeating sequence of digits.

Following on from the definitions 3.5 and 3.4 we might suppose that a definition such as the following suffices to define our Thirds set:

Definition 3.6. For $r \in \mathbb{R}, 0 \leq r \leq 1$

$$r \in Q \iff \neg \left\{ \exists m \in \mathbb{N}, \exists k \in \mathbb{N}, 0 \leq k \leq (3^{m-1} - 1) \wedge \frac{3k+1}{3^m} < r < \frac{3k+2}{3^m} \right\}$$

and its complement as:

Definition 3.7. For $r \in \mathbb{R}, 0 \leq r \leq 1$

$$r \in Q^c \iff \left\{ \exists m \in \mathbb{N}, \exists k \in \mathbb{N}, 0 \leq k \leq (3^{m-1} - 1) \wedge \frac{3k+1}{3^m} < r < \frac{3k+2}{3^m} \right\}$$

In order to establish the membership of Q , we can consider what remains of the interval $[0, 1]$. For any given n , there is a common denominator for the endpoints for that n , hence the right

endpoint of a removed interval and the left endpoint of the next removed interval for that n are the endpoints of a remaining closed interval as:

$$\left[\frac{2k}{3^n}, \frac{2k+1}{3^n} \right] \text{ with midpoint } \frac{4k+1}{2 \cdot 3^n}, \text{ where } 0 \leq k \leq n,$$

Some of such endpoints are included in the set Q but not all, since some are removed for some $m < n$. The midpoint of every such interval is the midpoint of the subsequent removal at $n + 1$ of the middle third of that interval. We now have a contradiction, since no such midpoint can be in the set 3.6, but at the same time, there is no state where the remaining intervals do not have a non-zero measure which means that there are always remaining intervals with such rational midpoints.

As noted in Section 3.2, the FRD points given by the repeated iteration are numbers that are all rational, and all have a non-terminating ternary expansion that consists only of the digits 0 and 2, which after some digit consists of an infinitely repeating sequence of digits. The calculation of the FRD points is based on the existence of intervals of non-zero measure that are defined by an iterative process which never completes, and which necessarily entails that there are always intervals of non-zero measure not removed by that process.

However, the conventional notion of the Thirds set as by the [Thirds Conjecture](#) is that it is a non-denumerable set that includes numbers that have a single ternary expansion and which consists only of the digits 0 and 2, and which have an infinite expansion that do not have a terminal infinitely repeating sequence of digits, i.e., they are irrational.

If it were the case that the Thirds set could be the result of infinitely many recursive iterative removals of intervals, then by the same reasoning, since every FRD point is associated with a specific iteration that corresponds to a specific natural number, then the only points in the set so defined, besides the rational interval endpoints, would be denumerable rational FRD points.

Hence there is a contradiction between the requirements of the [Thirds Conjecture](#) and any definition such as 3.6, since the definition does not define any such irrational points. This demonstrates that the naive definition of sets such as by the definitions 3.7 and 3.6 does not in fact reflect the intuitive notion that infinitely many iterations of the union of the intervals can result in the intervals somehow transitioning from being non-degenerate to being degenerate. There cannot be an actual transition from a finite number of iterations to infinitely many such iterations, but the definitions of 3.7 and 3.6 do not include anything that defines degenerate intervals. It is a fallacy to assume that such definitions are sufficient to establish set membership.

3.3 Defining the Thirds set

By using the definitions 3.7 and 3.6 above, except that we explicitly include a limiting condition, this gives definitions which unlike the previous cases are not first-order, giving the Thirds set as:

Definition 3.8. For $r \in \mathbb{R}, 0 \leq r \leq 1, m \in \mathbb{N}, m > 0, k \in \mathbb{N}$,

$$r \in \text{Thirds set} \iff [0, 1] - \lim_{m \rightarrow \infty} \bigcup \left\{ r : 0 \leq k \leq (3^{m-1}-1) \wedge \frac{3k+1}{3^m} < r < \frac{3k+2}{3^m} \right\}$$

and its complement as:

Definition 3.9. For $r \in \mathbb{R}, 0 \leq r \leq 1, m \in \mathbb{N}, m > 0, k \in \mathbb{N}$,

$$r \in \text{Thirds}^{\text{C}} \iff \lim_{m \rightarrow \infty} \bigcup \left\{ r : 0 \leq k \leq (3^{m-1} - 1) \wedge \frac{3k+1}{3^m} < r < \frac{3k+2}{3^m} \right\}$$

Unlike the previous case, there is no inherent contradiction in this definition. As for the previous case, the remaining intervals up to a given n are of the form:

$$\left[\frac{2k}{3^n}, \frac{2k+1}{3^n} \right] \text{ with midpoint } \frac{4k+1}{2 \cdot 3^n}, \text{ where } 0 \leq k \leq n,$$

and the midpoint of each such interval for a given n is a midpoint of an interval that is removed for $n+1$. As for the previous case, some of the endpoints of these defined intervals are included in the Thirds set but not all, since some are removed for some $m < n$.

Since these remaining intervals are closed intervals, then as for the case discussed in Section 2.1, the limit state is that the endpoints coincide, leaving single degenerate intervals. And in the same way as for that case, no such limit points can be numbers of the form $(4k+1)/(2 \cdot 3^n)$, otherwise there would be some n such that the point $(4k+1)/(2 \cdot 3^n)$ would be the midpoint of an interval that has, by definition, non-zero measure, rather than the midpoint/endpoint of a degenerate interval of zero measure.

Hence such limit points must either be the rational FRD points, or the irrational limit FRD points. Both are included since the overall limit state of the set includes points that cannot be achieved by recursive iterative removal of intervals. These points cannot have a digit 1 at any n^{th} digit since then it would either be an endpoint for some m^{th} interval, or else it would have been removed at some iteration. And because the limit FRD points do not have any associated m of the enumeration, it follows that while those points also have ternary expansions with only the digits 0 and 2, those points are not rational, but irrational.

Furthermore, these limit FRD points are not a denumerable set of points, since none of them are associated with an m of the enumeration. As in Sections 1 and 2.2, this does not indicate that there are “more” FRD points than there are points in the Thirds complement set. The non-denumerability is simply the result of the fact that it is not possible to define the details of a transition from enumerated intervals to limit state non-enumerated points since there is no such transition.

The above definition with explicit limiting conditions fulfils the requirements for the Thirds set as set out in the [Thirds Conjecture](#), being a non-denumerable set of the points as described by that conjecture: the rational endpoints, the rational FRD points, and the irrational limit FRD points.

In summary, the explicit inclusion of the limit condition removes all the paradoxes associated with the conventional accounts of the Thirds set, and shows that the paradoxes arise because of the intuitive assumption that the notion of infinitely many recursive iterations of the removal of middle thirds defines a specific set.

3.4 A set defined by infinitely many closed intervals

It is of interest to examine definitions that are the same as for the Thirds set definitions 3.9 and 3.8 above, except that the intervals removed are all closed intervals, giving definitions which again are not first-order:

Definition 3.10. For $r \in \mathbb{R}, 0 \leq r \leq 1, m \in \mathbb{N}, m > 0, k \in \mathbb{N}$,

$$r \in C \iff [0, 1] - \lim_{m \rightarrow \infty} \bigcup \left\{ r : 0 \leq k \leq (3^{m-1} - 1) \wedge \frac{3k+1}{3^m} \leq r \leq \frac{3k+2}{3^m} \right\}$$

and its complement:

Definition 3.11. For $r \in \mathbb{R}, 0 \leq r \leq 1, m \in \mathbb{N}, m > 0, k \in \mathbb{N}$,

$$r \in C^c \iff \lim_{m \rightarrow \infty} \bigcup \left\{ r : 0 \leq k \leq (3^{m-1} - 1) \wedge \frac{3k+1}{3^m} \leq r \leq \frac{3k+2}{3^m} \right\}$$

For these definitions, the remaining intervals for a given m are open intervals of the form:

$$\left(\frac{2k}{3^n}, \frac{2k+1}{3^n} \right)$$

and the limit state is that the endpoints coincide, which for open intervals, as for the case discussed in Section 2.2, is that the remaining intervals disappear, and hence there are no elements in the set C .

3.5 The measure of the Thirds set

It is said that the limiting sum of the intervals that are removed by the denumerable iterative removal of middle thirds is 1, and hence the measure of the complementary set to the Thirds set cannot be less than 1; neither can it be more than 1, hence the measure of the complementary set is taken to be 1, and from this, the measure of the Thirds set must be zero. However, this method of derivation of measure cannot be applied to the general case of Smith-Volterra-Cantor sets, as will be demonstrated in Section 4.1.

4 Smith-Volterra-Cantor sets

The Thirds set is a special case of the type of set that is commonly referred to as a Smith-Volterra-Cantor^E set^[10] In the same way as for the Thirds set, it is conventionally informally said that a Smith-Volterra-Cantor set is a set that results from infinitely many recursive iterations.

Smith-Volterra-Cantor sets are based on the notion that for finitely many iterations, at the n^{th} iteration an open interval of actual (as opposed to relative) measure $1/p^n$ is removed from the middle of every interval remaining from the previous iteration. For the Cantor Thirds set, the corresponding value of p is 3. If $p < 3$, the iterative process must terminate, as for some finite n the n^{th} iteration would require removal of a greater measure than that remaining (this is shown below). Hence for non-terminating iterations, $p \geq 3$ must apply.^F It can be observed that for the Thirds set, where $p=3$, that it is also the case that a *relative* measure of $1/p$ of the measure of every remaining interval is removed from the middle of that interval, but it must be noted that this observation applies only to the Thirds set where $p=3$, and it does not apply for any other value of p .

^E These sets are a subset of what are termed fat Cantor sets.^{[5],[Ch.5]}

^F Generally it is assumed that for Smith-Volterra-Cantor sets p is a natural number, but this does not have to be the case.

Formally, we can define the complement of the Smith-Volterra-Cantor set including a limiting condition as:

Definition 4.1. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}, n > 0, k \in \mathbb{N}$,

$$r \in SVC^c \iff \lim_{n \rightarrow \infty} \bigcup \left\{ r : 0 \leq k \leq (2^{n-1} - 1) \wedge A(n) < r < B(n) \right\}$$

and the Smith-Volterra-Cantor set as:

Definition 4.2. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}, n > 0, k \in \mathbb{N}$,

$$r \in SVC \iff [0, 1] - \lim_{n \rightarrow \infty} \bigcup \left\{ r : 0 \leq k \leq (2^{n-1} - 1) \wedge A(n) < r < B(n) \right\}$$

where the points $A(n)$ and $B(n)$ are definable in terms of $1/p^n$ (see [Appendix A](#)).

A commonly cited example of a Smith-Volterra-Cantor set is when $p=4$, where, in terms of iterations, at the first iteration, an actual measure of $1/4$ units is removed from the middle of the initial interval of unit measure. For the second iteration, an actual measure of $1/16$ units is removed from the middle of each interval that is left from the previous iteration. For the third iteration, an actual measure of $1/64$ units is removed from the middle of each interval that is left from the previous iteration. And so on.

In the same way as for the Thirds Set, the Smith-Volterra-Cantor set as defined by 4.2 consists entirely of isolated points, since it cannot include any non-degenerate intervals.

4.1 The measure of a Smith-Volterra-Cantor set

The conventional analysis of the measure of a Smith-Volterra-Cantor set is as follows:

Considering the general case, there are 2^{n-1} remaining intervals before the n^{th} iteration occurs. Since at the n^{th} iteration, an actual measure of $1/p^n$ is removed from each interval, then the total amount removed by the n^{th} iteration is:

$$\frac{2^{n-1}}{p^n} = \frac{1}{2} \left(\frac{2}{p} \right)^n$$

Therefore, after the n^{th} iteration, the total measure that has been removed is:

$$\frac{1}{2} \left\{ \frac{2}{p} + \left(\frac{2}{p} \right)^2 + \left(\frac{2}{p} \right)^3 + \dots + \left(\frac{2}{p} \right)^n \right\}$$

Since $\frac{2}{p} + \left(\frac{2}{p} \right)^2 + \left(\frac{2}{p} \right)^3 + \dots + \left(\frac{2}{p} \right)^n$ is a standard geometric series, the **limiting** value of the series as n increases is:

$$\frac{\frac{2}{p}}{1 - \frac{2}{p}} = \frac{2}{p-2}$$

provided $p > 2$. For non-terminating iterations $p \geq 3$, so this condition is satisfied. Hence the **limiting** value of the total measure removed, according to this method of calculation, is

half of this, which is $\frac{1}{p-2} \cdot G$.

According to this method of calculation, for $p=3$, the limiting value of the total measure removed is 1, for $p=4$ the limiting value of the total measure removed is $1/2$, for $p=5$ the limiting value of the total measure removed is $2/3$, and so on. According to this method of calculation, by choosing an appropriate real number value for p , any value between 0 and 1 can be given for the limiting value of the total measure removed. Furthermore, for all $p \geq 3$ the limiting measure of the measure of a removed interval as n increases is zero, according to this method of calculation, since the actual measure of a removed interval is $1/p^n$.

For $p=4$, the conventional reading is that the recursive iterations define a set where the *limit* of the sum of the measures that have been removed is $1/2$. Since the limit of the sum of the intervals removed is $1/2$, then, according to this method of calculation, the total measure of what remains in the Smith-Volterra-Cantor set for $p=4$ must be $1 - 1/2$, which is $1/2$.

It can be noted that although most authors use the approach as indicated above, at least one author employs a different method of calculating the measure of an Smith-Volterra-Cantor set. Nelson^[8] asserts that the outer measure of the Thirds set is given by the following consideration:

The set given by the n^{th} iteration, C_n , consists of 2^n intervals of measure $1/3^n$, so the outer measure

$$m^*(C) \leq 2^n \cdot \frac{1}{3^n} = \left(\frac{2}{3}\right)^n$$

and the only way this can hold for every positive integer n is for $m^*(C) = 0$.

Note that both the above methods of calculating measure use the notion of a limit. The difference in these two different approaches outlined above can be summarized as follows:

4.2 Total measure less limiting value of summation of removed measures

Let A_n be the set of all endpoints of removed intervals that are defined up to and including the n^{th} iteration. This is a denumerable set and a function can readily be defined that includes the elements of this set. For the n^{th} iteration, there are

$$s = \sum_{i=1}^n 2^i$$

endpoints, so there are s points in the set A_n . From the recursive definition of the iterative removal of intervals (see [Appendix A](#)), a function $f(m)$ can be readily defined that enumerates the elements of the set A_n , where $1 \leq m \leq s$, such that $f(m)$ gives the m^{th} largest point of the set A_n . Then we can define the sum of the measures of all removed intervals up to and including the n^{th} iteration as:

$$\sum_{i=1}^{n/2} \{f(2i) - f(2i-1)\}$$

^G From this it follows that for $2 < p < 3$, since the value of $\frac{1}{p-2}$ is greater than 1, the iteration must terminate at some finite n .

The limiting value of the measures of all removed intervals is thus

$$\lim_{i \rightarrow \infty} \sum_{i=1} \{f(2i) - f(2i-1)\}$$

The conventional analysis of the measure makes the *assumption* that:

$$\lim_{n \rightarrow \infty} \sum_{n=1} \frac{1}{2} \left(\frac{2}{p}\right)^n \stackrel{???}{=} \lim_{i \rightarrow \infty} \sum_{i=1} \left\{ \lim_{i \rightarrow \infty} \{f(2i) - f(2i-1)\} \right\} \quad (4.2.1)$$

where we use the double limit expression to indicate the limit of the summation of the limit measure of the intervals as $i \rightarrow \infty$, which for $p = 4$ gives:

$$\lim_{n \rightarrow \infty} \sum_{n=1} \frac{1}{2} \left(\frac{2}{4}\right)^n = \frac{1}{2} \stackrel{???}{=} \lim_{i \rightarrow \infty} \sum_{i=1} \left\{ \lim_{i \rightarrow \infty} \{f(2i) - f(2i-1)\} \right\}$$

so that the measure of the Smith-Volterra-Cantor set under the assumption in 4.2.1 is $1/2$.

4.3 Limiting value of summation of remaining measures

We now let B_n be the set A_n together with the points 0 and 1, i.e., the set of all endpoints of removed intervals that are defined up to and including the n^{th} iteration, along with the points 0 and 1. For the n^{th} iteration, there are

$$t = 2 + \sum_{i=1}^n 2^i$$

points that include the endpoints and 0 and 1, so there are t points in the set B_n . As for the function $f(m)$ for the set A_n , a function $g(m)$ can readily be defined that enumerates the elements of this set B_n , where $1 \leq m \leq t$, such that $g(m)$ gives the m^{th} largest point of the set B_n . Then we can define the sum of the measures of all *remaining* intervals after the n^{th} iteration as:

$$\sum_{i=1}^{n/2} \{g(2i) - g(2i-1)\}$$

Therefore, the limiting value of the summation of the limiting values of the measures of all remaining intervals as $n \rightarrow \infty$ is:

$$\lim_{i \rightarrow \infty} \sum_{i=1} \left\{ \lim_{i \rightarrow \infty} \{g(2i) - g(2i-1)\} \right\}$$

In the case of the sets of removed intervals, the intervals already removed in a set A_n always appear in any subsequent set A_m , where $m > n$. However, this is not the case for the sets B_n of remaining intervals, and at each iteration the measure of *every* interval that remains is always smaller than on the previous iteration, and since there is no lower limit to the measures of the remaining intervals, we have, regardless of the value of p :

$$\lim_{i \rightarrow \infty} \sum_{i=1} \left\{ \lim_{i \rightarrow \infty} \{g(2i) - g(2i-1)\} \right\} = 0$$

This result contradicts the previous result in Section 4.2 above.

4.4 The source of the discrepancy of measure

The above demonstrates that the attempt to calculate a measure by a naive application of a limiting condition may not ensure a correct result, particularly if unwarranted assumptions are made. Furthermore, it was shown in Sections 3.4 and 3.3 that the conventional assumption that a valid definition of the Thirds set (and in general any Smith-Volterra-Cantor set) is in terms of an infinite number of recursive removals of intervals, without any reference to limit states, is incorrect. It follows that the naive assumption that one can apply a *limit* state to the *measures* of removed intervals by:

$$\lim_{n \rightarrow \infty} \sum_{n=1} = \frac{1}{2} \left\{ \frac{2}{p} + \left(\frac{2}{p}\right)^2 + \left(\frac{2}{p}\right)^3 + \dots + \left(\frac{2}{p}\right)^n \right\} \quad (4.4.1)$$

(as detailed above in 4.2.1), while at the same time ignoring the constitution of the actual set that results from the application of the limit condition to those intervals lacks any logical foundation.

The conventional account claims that the measure of the set of removed intervals over a unit interval can be treated by a consideration of a value A given by a limit condition on a set of denumerable intervals, which, by the subtraction of this value A from unity, gives a value as the measure of a set the non-denumerable isolated points.

Although it is the case for any given iteration that there is a set of rational endpoints of removed intervals that is denumerable (and all rational), since a limit condition is required to give a valid definition of the resultant set, that set of endpoints of the removed complete intervals is necessarily a non-denumerable set (and not all rational). It follows that the defined resultant set of removed complete intervals is not denumerable with regard to measure, since if that were the case:

1. there must be a denumerable point and an associated determinable measure for each complete removed interval, and
2. both endpoints of each interval must be denumerable.

Either of these implies the other, but neither applies in this case.

The fact is that equation 4.4.1 applies to the simple case of non-overlapping intervals with denumerable endpoints, but a common justification for its use in the case for Smith-Volterra-Cantor sets is by referring to such cases. An example is the summation of the measure of the intervals $(1/2^{n+1}, 1/2^n)$, where the limit of the summation gives the correct result when the left endpoint of one interval is the right endpoint of the interval in the subsequent interval, and where there is a limit point at zero. It involves no irrational endpoints and there is no non-denumerable set of endpoints. Such cases are special cases, and it is a logical error to generalize from a small number of specific cases with certain characteristics to general cases that do not have those characteristics.

The naive assumption that such simple cases justify the application of the same method to more complex cases involving non-denumerable sets of limit points has no logical basis, and the conventional conclusion ignores the infinite set of non-denumerable limit points of the resultant intervals to give a simplistically erroneous value of measure, while at the same time invoking the non-denumerability of those very same limit points to justify that result by claiming that the result is due to that set of non-denumerable points.

Compounding that fallacy, there is also the assumption that an infinite number of numbers can be summated, which assumes that infinitely many iterations of an additive algorithm can be completed; multiplication is merely repeated summation. But any value that is given as the value of the summation of infinitely many values is in fact *always* given by consideration of the limiting value, not an actual summation of infinitely many values. The actual formula that is used in this case is obtained by calculating the *limiting* case of the sum of a geometric series. There is a failure to recognize that there cannot be any actual transition from any large finite set of entities to an infinitely large set of entities, or from an interval which covers infinitely many points to an interval that has only one point, or to no interval at all (if there were, there would be some finite number denoting the iteration at which such a transition occurs), which is why the correct analysis of such situations must correctly analyze all the limit states that are involved.

There is also an assumption that the measures $1/p, 1/p^2, 1/p^3 \dots$ are independent of the intervals that give rise to these measures. But without an assumption that each such measure has an independent existence, then it is evident that each measure is defined by a left endpoint and a right endpoint. The assumption is that one can ignore the origin of the terms in the series $1/p, 1/p^2, 1/p^3 \dots$, whereas this series is derived from the defined endpoints, and hence is derived from the series:

$$r_1 - l_1 + r_2 - l_2 + r_3 - l_3 + \dots$$

where l_1 and r_1 are the left and right endpoints of the first interval, l_2 and r_2 are the left and right endpoints of the second interval, l_3 and r_3 are the left and right endpoints of the third interval, and so on. For a finite sum, it is completely immaterial as to which order the additions and subtractions are performed. But where the process continues infinitely, and where the intervals are not adjacent, it is a completely different matter. It is well known that for an infinite alternating series which has both positive and negative terms, and which is not absolutely convergent the limiting sum is dependent on the order in which the terms appear in the series. The conventional assumption, as in 4.2.1 above, completely ignores this crucial detail. Furthermore, while a specific order is assumed for the series, in fact there is no reason why the same intervals could not be defined in infinitely many different orderings.^H

The crucial point that must be addressed in any attempt to apply the notion of a limiting value of a measure is the analysis as to whether the notion of a limiting value can be sensibly applied to a given case. It is fallacious to assume that an equation such as 4.4.1 which involves a sequence of intervals that gives a resultant denumerable set of intervals where there is only one instance of a limit to zero, must apply to a case where there is a resultant non-denumerable set of intervals where there are infinitely many limits to zero.

^HThe two endpoints of any given interval do not need to appear consecutively in any such ordering. For example, if $l_1, r_1, l_2, r_2, l_3, r_3, l_4, r_4$ are in ascending order, then the total measure of the intervals $(l_1, r_1), (l_2, r_2), (l_3, r_3), (l_4, r_4)$ can be given by $r_4 - l_1 + r_3 - l_2 + r_1 - l_2 + r_2 - l_3 + r_3 - l_4$. For degenerate single point limit state closed intervals, the left endpoints and right endpoints are identical, hence appear twice in a summation, and the order in which they appear affects the limit state of the summation.

Appendix A: Endpoints of general case of removal of intervals

We can give a recursive definition that will give the values of the endpoints for any iteration of the removal of the open middle section of existing intervals, for any real number value of $p \geq 3$, where the actual measure of $1/p^n$ is removed from the middle of each interval. In the following definition, the L points are the left endpoints of the remaining intervals, and the R points are the right endpoints of the remaining intervals, see the diagram below.^I

Definition A.3.

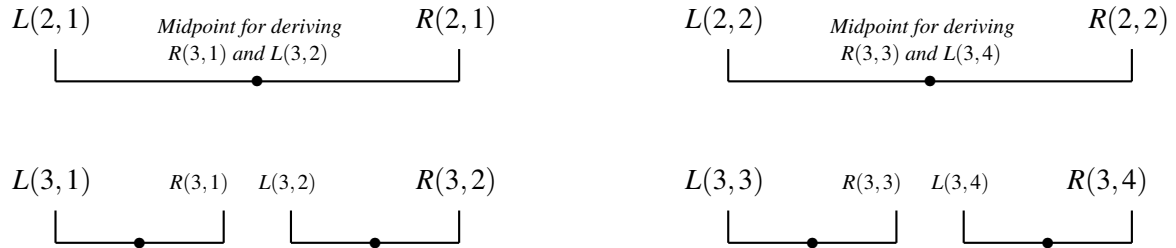
$$\text{For } n = 0, k = 0, \quad L(n, k) = 0, \quad R(n, k) = 1 \quad (\text{A.1.a})$$

$$\text{otherwise, for } n \in \mathbb{N}, n \geq 1, k \in \mathbb{N}, 1 \leq k \leq 2^{n-1}$$

$$L(n, k) = \begin{cases} 0 & \text{if } k = 1 & (\text{A.1.b}) \\ L\left(n-1, \frac{k+1}{2}\right) & \text{if } k \text{ odd, } k \neq 1 & (\text{A.1.c}) \\ \frac{1}{2} \left[L\left(n-1, \frac{k}{2}\right) + R\left(n-1, \frac{k}{2}\right) + \frac{1}{p^{n-1}} \right] & \text{if } k \text{ even} & (\text{A.1.d}) \end{cases}$$

$$R(n, k) = \begin{cases} 1 & \text{if } k = 2^{n-1} & (\text{A.1.e}) \\ R\left(n-1, \frac{k}{2}\right) & \text{if } k \text{ even, } k \neq 2^{n-1} & (\text{A.1.f}) \\ \frac{1}{2} \left[L\left(n-1, \frac{k+1}{2}\right) + R\left(n-1, \frac{k+1}{2}\right) - \frac{1}{p^{n-1}} \right] & \text{if } k \text{ odd} & (\text{A.1.g}) \end{cases}$$

$L(1, 1)$ *Midpoint for deriving* $R(1, 1)$
 $R(2, 1)$ and $L(2, 2)$



In general for a given n there are $k = 2^n$ endpoints, so there are $k = 2^{n-1}$ left endpoints and $k = 2^{n-1}$ right endpoints, hence $1 \leq k \leq 2^{n-1}$. $L(n, k)$ and $R(n, k)$ indicate the left and right

^I The definition is somewhat easier when done in this manner, although the endpoints could be defined as the left and right endpoints of the intervals removed at the n^{th} iteration.

endpoints respectively of the remaining intervals for the n^{th} iteration. Case A.1.b ensures that the leftmost endpoint is always 0, and similarly, case A.1.e ensures that the rightmost endpoint is always 1. At each iteration, the number of endpoints doubles, and so the maximum value of k doubles at each iteration. Half of the endpoints for a given n are given by the endpoints existing before the iteration, and the other half is given by newly formed endpoints. The left unaltered endpoints always have an even k before the iteration, and an odd k after the iteration, hence the endpoint is carried over to the next value of n by case A.1.c by $\frac{k+1}{2}$; the right unaltered endpoints always have an odd k before the iteration, and an even k after the iteration, hence the endpoint is carried over to the next value of n by case A.1.f by $k/2$ (see the diagram below).

New endpoints for the n^{th} iteration are given by cases A.1.d and A.1.g. The midpoint of an interval remaining from the previous iteration is determined from the endpoints of that interval, i.e., $\frac{1}{2}[L(\) + R(\)]$, and the new endpoints are given by subtracting and adding the appropriate value to that midpoint, i.e., $\frac{1}{2p^{n-1}}$. New left endpoints always have an even k , and new right endpoints always have an odd k . The reason why $k+1$ appears in the definition for case A.1.g, while k appears in the corresponding case A.1.d is that, for a given k and a given n , the $L(n,k)$ and the $R(n,k)$ are defined in terms of the midpoint of a previous remaining interval, which itself is defined in terms of the left and right endpoints of that interval; hence the new left endpoint has a k that is 1 greater than the new right endpoint for a given previous remaining interval. The diagram above illustrates how this operates.

Having defined the endpoints, we can now define the complement of the Smith-Volterra-Cantor set for any given p .

Definition A.4. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}, n > 0, k \in \mathbb{N}$,

$$r \in SVC^c \iff \lim_{n \rightarrow \infty} \bigcup \left\{ r : 0 \leq k \leq 2^{n-1} \wedge R(n,k) < r < L(n,k+1) \right\}$$

and the Smith-Volterra-Cantor set as:

Definition A.5. For $r \in \mathbb{R}, 0 \leq r \leq 1, n \in \mathbb{N}, n > 0, k \in \mathbb{N}$,

$$r \in SVC \iff [0, 1] - \lim_{n \rightarrow \infty} \bigcup \left\{ r : 0 \leq k \leq 2^{n-1} \wedge R(n,k) < r < L(n,k+1) \right\}$$

The problems noted earlier that arises from conventional considerations of the iterative recursion, such problems are obviated when the definition is given in the above manner, which is that a real number r is defined as being an element of a set if, for a given proposition with n as a free variable, there exists an n such that the proposition is satisfied.

While this definition of an Smith-Volterra-Cantor set is derived from the notion of an iterative process, the definition itself does not require such a process nor the completion of such a process. While it is not proven that a Smith-Volterra-Cantor set can only be defined in terms such as the above, it appears unlikely that there could be a finite definition that does not involve some sort of recursive reference.

If we examine a logical definition of the Smith-Volterra-Cantor set as given by A.5 above, we see that the Smith-Volterra-Cantor set is defined as a set of isolated points that are defined such that there is no lower limit to the measure of the interval between $L(n,k)$ and $R(n,k)$ as $n \rightarrow \infty$. It also follows that there is no lower limit to the measure that an interval between $R(n,k)$ and

$L(n, k + 1)$ can be as $n \rightarrow \infty$. The limit state is that the intervals between $R(n, k)$ and $L(n, k + 1)$ is that the intervals disappear (for closed intervals, the limit state is degenerate point intervals).

Appendix B: A non-denumerable set of points

Conventionally, the elements of Thirds set are numbers that have a ternary expansion that consists only of the digits 0 and 2 as described above in the [Thirds Conjecture](#).

The elements of the set so described can be mapped to numbers in the interval $(0, 1)$ by replacing each digit 2 by a digit 1, which gives a one-to-one correspondence, in terms of binary expansions, of all points in the set to a subset S of numbers in the interval $(0, 1)$. All numbers with dual ternary expansions are rationals that have a finite ternary expansion, and so have the expansions of the form $0.xxx\dots222\dots$ or $0.xxx\dots2$ where x is 0 or 2.^J This means that the only numbers that are not included in the binary expansions of $(0, 1)$ by the correspondence to S are numbers of the form $0.xxx\dots111\dots$ or $0.xxx\dots1$, where x is 0 or 1; these constitute a denumerable set of rationals. Hence it follows that the set S of binary expansions is not denumerable, and that the set defined that is in accordance with the [Thirds Conjecture](#) is also not denumerable.^K

Appendix C: An enumeration of the rationals

A typical enumeration of the rationals is: $\frac{1}{2}, \frac{1}{3}, \frac{2}{3}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \dots$. This can be defined in terms of an algorithm which defines the n^{th} rational for any given n , viz.:

1. If $\sqrt{1 + 8n}$ is a Natural number, then:
2. $m = \frac{-1 + \sqrt{1 + 8n}}{2}$ and the n^{th} rational is $\frac{m}{m + 1}$
3. Otherwise, $\sqrt{1 + 8n}$ is not a Natural number, so let $t = 1$:
4. If $\sqrt{1 + 8(n + t)}$ is a Natural number, then:
5. $m = \frac{-1 + \sqrt{1 + 8(n + t)}}{2}$ and the n^{th} rational is $\frac{m - t}{m + 1}$
6. Otherwise let $t = t + 1$ and repeat from step 4.

^J These are the points $\frac{3k+1}{3^m}$ and $\frac{3k+2}{3^m}$ in the definitions 3.9 and 3.8.

^K See also "On the Density of Linear Sets of Points" by W. H. Young^[15] re this aspect.

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